M-math 2nd year Mid Semester Exam
Subject : Advanced Probability

Time : 3.00 hours | Max.Marks 50.

1. Let $\mu_n, n \geq 1$ and μ be finite measures on $\mathbb{R}$. Suppose that $\mu_n(a,b] \to \mu(a,b]$ for all $a < b$ with $\mu\{a\} = \mu\{b\} = 0$ i.e. the sequence $\{\mu_n\}$ converges to μ vaguely. Suppose further that that

$$\sup_{n \geq 1} \mu_n(\mathbb{R}) < \infty.$$

Then show that

$$\int f d\mu_n \to \int f d\mu$$

for all $f : \mathbb{R} \to \mathbb{R}$ continuous and vanishing at ∞. Hint : choose an interval $(a,b]$ such that μ_n restricted to this interval and normalised yields probability measures. (15)

2. State the inversion formula for characteristic functions on $\mathbb{R}^2$ and prove your result. (6+9)

3. Let E be a complete separable metric space and μ a probability measure on its Borel sets. Show that μ is tight. (10)

4. Let $\{X_i; i \geq 0\}$ be a sequence of i.i.d random variables and let N be a Poisson r.v. with parameter 1, independent of $\{X_i; i \geq 0\}$. Let $Y := X_0 + X_1 + \cdots + X_N$. Show that Y is an infinitely divisible random variable. (10)

5. Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence with $E|X_1| < \infty$. Let $Y_n := X_n|_{\{|X_n| \leq 1\}}$. Show that

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{n}$$

converges almost surely. (10)